

# CHARACTERS OF REPRESENTATIONS OF AFFINE KAC-MOODY LIE ALGEBRAS AT THE CRITICAL LEVEL

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## 1. INTRODUCTION AND MAIN RESULTS

1.1. Let  $\bar{\mathfrak{g}}$  be a complex simple Lie algebra of rank  $l$ ,  $\mathfrak{g}$  non-twisted affine Kac-Moody Lie algebra associated with  $\bar{\mathfrak{g}}$ :

$$(1) \quad \mathfrak{g} = \bar{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D.$$

The commutation relations of  $\mathfrak{g}$  are given by the following.

$$\begin{aligned} [X(m), Y(n)] &= [X, Y](m+n) + m\delta_{m+n,0}(X|Y)K, \\ [D, X(m)] &= mX(m), \quad [K, \mathfrak{g}] = 0 \end{aligned}$$

for  $X, Y \in \bar{\mathfrak{g}}$ ,  $m, n \in \mathbb{Z}$ , where  $X(m) = X \otimes t^m$  with  $X \in \bar{\mathfrak{g}}$  and  $m \in \mathbb{Z}$  and  $(\cdot|\cdot)$  is the normalized invariant inner product of  $\bar{\mathfrak{g}}$ . We identify  $\bar{\mathfrak{g}}$  with  $\bar{\mathfrak{g}} \otimes \mathbb{C} \subset \mathfrak{g}$ . Fix the triangular decomposition  $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_+$ , and the Cartan subalgebra of  $\mathfrak{g}$  as  $\mathfrak{h} = \bar{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}D$ . We have  $\mathfrak{h}^* = \bar{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$ , where  $\Lambda_0$  and  $\delta$  are elements dual to  $K$  and  $D$ , respectively.

Let  $L(\lambda)$  be the irreducible highest weight representation of  $\mathfrak{g}$  of highest weight  $\lambda \in \mathfrak{h}^*$  with respect to the standard triangular decomposition  $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ , where

$$\mathfrak{n}_- = \bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{g}} \otimes \mathbb{C}[t^{-1}]t^{-1}, \quad \mathfrak{n}_+ = \bar{\mathfrak{n}}_+ \oplus \bar{\mathfrak{g}} \otimes \mathbb{C}[t]t.$$

The central element  $K$  acts on  $L(\lambda)$  as the multiplication by the constant  $\langle \lambda, K \rangle$ , which is called the *level* of  $L(\lambda)$ . The level  $\langle \lambda, K \rangle = -h^\vee$  is called *critical*, where  $h^\vee$  is the dual Coxeter number of  $\bar{\mathfrak{g}}$ .

1.2. Let  $\text{ch } L(\lambda)$  be the formal character of  $L(\lambda)$ :

$$\text{ch } L(\lambda) = \sum_{\mu \in \mathfrak{h}^*} e^\mu \dim_{\mathbb{C}} L(\lambda)^\mu,$$

where  $L(\lambda)^\mu$  is the weight space of  $L(\lambda)$  of weight  $\mu$ .

The Weyl-Kac character formula [Kac74] gives an explicit formula of  $\text{ch } L(\lambda)$  in the case that  $L(\lambda)$  is an integrable representations of  $\mathfrak{g}$ . It is known that Kac-Wakimoto admissible representations [KW88, KW89] also have the Weyl-Kac type character formulas. The celebrated Kazhdan-Lusztig conjecture [KL79] (proved by [BB81, BK81]) has been generalized to  $\mathfrak{g}$  by Kashiwara-Tanisaki [KT95, KT96, KT98, KT00] and Casian [Cas96]. As a result the character  $\text{ch } L(\lambda)$  is known for *any*  $L(\lambda)$  provided that its level is *not* critical (see [KT00] for the most general formula).

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1.3. On the contrary not much is known about the characters of  $L(\lambda)$  at the critical level. It seems that only the *generic case* is known, that is, the case that  $\lambda$  satisfies the condition that

$$(2) \quad \langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{N} \quad \text{for all } \alpha \in \Delta_+^{\text{re}},$$

where  $\Delta_+^{\text{re}}$  is the set of positive real roots of  $\mathfrak{g}$ ,  $\rho = \bar{\rho} + h^\vee \Lambda_0$ ,  $\bar{\rho} = 1/2 \sum_{\alpha \in \bar{\Delta}_+} \alpha$  and  $\bar{\Delta}_+ \subset \Delta_+^{\text{re}}$  is the set of positive roots of  $\bar{\mathfrak{g}}$ . In this case the Kac-Kazhdan conjecture [KK79] (which is a theorem proved by [Hay88, GW89, FF88, Ku89]<sup>1</sup>) gives the following character formula of  $L(\lambda)$ :

$$(3) \quad \text{ch } L(\lambda) = \frac{e^\lambda}{\prod_{\alpha \in \Delta_+^{\text{re}}} (1 - e^{-\alpha})}.$$

By the existence of the Wakimoto modules at the critical level [Wak86, FF90b, Fre05], it follows that the irreducible representation  $L(\lambda)$  at the critical level in general has a character equal to or smaller than the right hand side of (3).

1.4. In this paper we study the irreducible highest weight representations of  $\mathfrak{g}$  at the critical level which are integrable over  $\bar{\mathfrak{g}}$ . Denote by  $\bar{\lambda}$  the restriction of  $\lambda \in \mathfrak{h}^*$  to  $\bar{\mathfrak{h}}$ . Set

$$P_{\text{crit}}^+ = \{\lambda \in \mathfrak{h}^*; \bar{\lambda} \in \bar{P}^+, \langle \lambda, K \rangle = -h^\vee\},$$

where  $\bar{P}^+$  the set of integral dominant weights of  $\bar{\mathfrak{g}}$ :

$$(4) \quad \bar{P}^+ = \{\bar{\lambda} \in \bar{\mathfrak{h}}^*; \langle \bar{\lambda}, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in \bar{\Delta}_+\}.$$

The  $L(\lambda)$  at the critical level is integrable over  $\bar{\mathfrak{g}}$  if and only if  $\lambda$  belongs to  $P_{\text{crit}}^+$ .

We have the following result.

**Theorem 1.** *Let  $\lambda \in P_{\text{crit}}^+$ . The character of  $L(\lambda)$  is given by*

$$\text{ch } L(\lambda) = \frac{\sum_{w \in \bar{W}} (-1)^{\ell(w)} e^{w \circ \lambda}}{\prod_{\alpha \in \bar{\Delta}_+} (1 - q^{-\langle \lambda + \rho, \alpha^\vee \rangle}) \prod_{\alpha \in \Delta_+^{\text{re}}} (1 - e^{-\alpha})},$$

where  $q = e^\delta$ ,  $\bar{W}$  is the Weyl group of  $\bar{\mathfrak{g}}$ ,  $\ell(w)$  is the length of  $w$  and  $w \circ \lambda = w(\lambda + \rho) - \rho$ .

1.5. Recently the representations of  $\mathfrak{g}$  at the critical level have been studied in detail by Frenkel and Gaitsgory [FG04, FG05, FG06, FG07] in the view point of the *geometric Langlands program*. Our original motivation was to confirm<sup>2</sup> Conjecture 5 of [Fre06] in the case that the *opers* are “graded” (see Theorem 10) by applying the method of the *quantum Drinfeld-Sokolov reduction* [FF90a, FF92, FBZ04] (cf. [FKW92, KRW03, KW04, Ara04, Ara05, Ara06, Ara07a]). Theorem 1 has been obtained as a byproduct of the proof.

<sup>1</sup>See also [Mat03, Fre05, Ara06].

<sup>2</sup>After finishing this paper we were notified that Frenkel and Gaitsgory have recently proved Conjecture 5 of [Fre06] and that Theorem 1 was known to E. Frenkel.

## 2. ENDMORPHISM RINGS, DUALITY AND THE CHARACTER FORMULA

**2.1. The category  $\mathcal{O}_{\text{crit}}^{\text{KL}}$ .** Denote by  $\mathcal{O}_{\text{crit}}^{\text{KL}}$  the full subcategory of the category of  $\mathfrak{g}$ -modules consisting of objects  $M$  such that the following hold: (1)  $K$  acts on  $M$  as the multiplication by  $-h^\vee$ , (2)  $D$  acts on  $M$  semisimply:  $M = \bigoplus_{d \in \mathbb{C}} M_d$ , where  $M_d = \{m \in M; Dm = dm\}$ , (3)  $\dim M_d < \infty$  for all  $d$ , (4) there exists a finite subset  $\{d_1, \dots, d_r\}$  of  $\mathbb{C}$  such that  $M_d = 0$  unless  $d \in \bigcup_{i=1}^r d_i - \mathbb{Z}_{\geq 0}$ .

Any object  $M$  of  $\mathcal{O}_{\text{crit}}^{\text{KL}}$  admits a weight space decomposition:  $M = \bigoplus_{\lambda} M^\lambda$ ,  $M^\lambda = \{m \in M; hm = \langle \lambda, h \rangle m \forall h \in \mathfrak{h}\}$ . We set  $\text{ch } M = \sum_{\lambda \in \mathfrak{h}^*} e^\lambda \dim M^\lambda$ .

The *Weyl module*

$$(5) \quad V(\lambda) = U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{g}} \otimes \mathbb{C}[t] \oplus \mathbb{C}K)} E(\lambda)$$

with  $\lambda \in P_{\text{crit}}^+$  belongs to  $\mathcal{O}_{\text{crit}}^{\text{KL}}$ . Here  $E(\lambda)$  is the irreducible finite-dimensional representation of  $\bar{\mathfrak{g}}$  of highest weight  $\bar{\lambda}$ , considered as a  $\bar{\mathfrak{g}} \otimes \mathbb{C}[t] \oplus \mathbb{C}K \oplus \mathbb{C}D$ -module on which  $\bar{\mathfrak{g}} \otimes \mathbb{C}[t]t$  acts by zero, and  $K$  and  $D$  act as the multiplication by  $-h^\vee$  and  $\langle \lambda, D \rangle$ , respectively. By the Weyl character formula one has

$$(6) \quad \text{ch } V(\lambda) = \frac{\sum_{w \in \bar{W}} (-1)^{\ell(w)} e^{w \circ \lambda}}{\prod_{j \geq 1} (1 - q^{-j})^\ell \prod_{\alpha \in \Delta_+^{\text{re}}} (1 - e^{-\alpha})}.$$

The  $V(\lambda)$  has  $L(\lambda)$  as its unique simple quotient.

**2.2. The derived algebra  $\mathfrak{g}'$  of  $\mathfrak{g}$ .** Let  $\mathfrak{g}'$  be the derived algebra of  $\mathfrak{g}$ :

$$(7) \quad \mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}] = \bar{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K.$$

One sees that each  $L(\lambda)$  remains irreducible over  $\mathfrak{g}'$ .

**2.3. The vertex algebra associated with  $\bar{\mathfrak{g}}$  at the critical level.** The vacuum Weyl module

$$(8) \quad \mathbb{V}_{\mathfrak{g}, \text{crit}} := V(-h^\vee \Lambda_0)$$

has the natural structure of vertex algebras (see eg. [Kac98, FBZ04]). The  $\mathbb{V}_{\mathfrak{g}, \text{crit}}$  is called the *universal affine vertex algebra associated with  $\bar{\mathfrak{g}}$  at the critical level*. Each object of  $\mathcal{O}_{\text{crit}}^{\text{KL}}$  can be regarded as a  $\mathbb{V}_{\mathfrak{g}, \text{crit}}$ -module.

For a vertex algebra  $V$  in general we denote by

$$(9) \quad Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1} \in (\text{End } V)[[z, z^{-1}]]$$

the quantum field corresponding to  $a \in V$ . Also, we write  $\text{Zh}(V)$  for the *Zhu algebra* [FZ92, Zhu96] (see also [NT05]) of  $V$ . One knows by [FZ92] that there is a natural isomorphism

$$(10) \quad \text{Zh}(\mathbb{V}_{\mathfrak{g}, \text{crit}}) \cong U(\bar{\mathfrak{g}}).$$

**2.4. Feigin-Frenkel's theorem.** The vertex algebra  $\mathbb{V}_{\mathfrak{g}, \text{crit}}$  has a large center [Hay88, FF92], which we denote by  $\mathfrak{Z}(\mathbb{V}_{\mathfrak{g}, \text{crit}})$ :

$$\mathfrak{Z}(\mathbb{V}_{\mathfrak{g}, \text{crit}}) = \{a \in \mathbb{V}_{\mathfrak{g}, \text{crit}}; a_{(n)}v = 0 \text{ for all } n \in \mathbb{Z}_{\geq 0}, v \in \mathbb{V}_{\mathfrak{g}, \text{crit}}\}.$$

Then

$$\mathfrak{Z}(\mathbb{V}_{\mathfrak{g}, \text{crit}}) = \{a \in \mathbb{V}_{\mathfrak{g}, \text{crit}}; [a_{(m)}, v_{(n)}] = 0 \text{ for all } m, n \in \mathbb{Z}, v \in \mathbb{V}_{\mathfrak{g}, \text{crit}}\}.$$

Let  $a \in \mathfrak{Z}(\mathbb{V}_{\mathfrak{g}, \text{crit}})$  and  $n \in \mathbb{Z}$ . The action of  $a_{(n)}$  on  $M \in \mathcal{O}_{\text{crit}}^{\text{KL}}$  commutes with the action of  $\mathfrak{g}'$ . Therefore each  $a_{(n)}$  acts as the multiplication by a constant on  $L(\lambda)$ .

Let  $I = \{1, 2, \dots, l\}$  ( $l = \text{rank } \bar{\mathfrak{g}}$ ),  $\{d_i; i \in I\}$  the exponents of  $\bar{\mathfrak{g}}$ ,  $\mathcal{Z}(\bar{\mathfrak{g}})$  the center of the universal enveloping algebra  $U(\bar{\mathfrak{g}})$  of  $\bar{\mathfrak{g}}$ .

There is a remarkable realization of  $\mathfrak{Z}(\mathbb{V}_{\mathfrak{g}, \text{crit}})$  due to Feigin and Frenkel [FF92] as a chiralization of Kostant's *Whittaker model* [Kos78] of  $\mathcal{Z}(\bar{\mathfrak{g}})$  (for details, see [FBZ04]). As a result one has the following description of  $\mathfrak{Z}(\mathbb{V}_{\mathfrak{g}, \text{crit}})$ .

**Theorem 2** (E. Frenkel and B. Feigin [FF92]). *There exist homogeneous vectors  $p^i \in \mathfrak{Z}(\mathbb{V}_{\mathfrak{g}, \text{crit}}) \cap (\mathbb{V}_{\mathfrak{g}, \text{crit}})_{-d_i-1}$  with  $i \in I$  that generate a PBW basis of  $\mathfrak{Z}(\mathbb{V}_{\mathfrak{g}, \text{crit}})$ : that is, there is a linear isomorphism*

$$\begin{array}{ccc} \mathbb{C}[p_{(-n)}^i; i \in I, n \in \mathbb{Z}_{\geq 1}] & \xrightarrow{\sim} & \mathfrak{Z}(\mathbb{V}_{\mathfrak{g}, \text{crit}}). \\ a & \mapsto & a|0\rangle, \end{array}$$

where  $|0\rangle$  is the highest weight vector of  $\mathbb{V}_{\mathfrak{g}, \text{crit}}$ .

**2.5. The linkage principle for  $\mathcal{O}_{\text{crit}}^{\text{KL}}$ .** Let  $p^i$  with  $i \in I$  be generators of  $\mathfrak{Z}(\mathbb{V}_{\mathfrak{g}, \text{crit}})$  as in Theorem 2. (We have  $p^i = p_{(-1)}^i|0\rangle$ .) Write

$$(11) \quad Y(p^i, z) = \sum_{n \in \mathbb{Z}} p_n^i z^{-n-d_i-1},$$

so that

$$(12) \quad [D, p_n^i] = np_n^i$$

on  $M \in \mathcal{O}_{\text{crit}}^{\text{KL}}$ . Set

$$(13) \quad R_{\mathcal{Z}} = \mathbb{C}[p_n^i; i \in I, n \in \mathbb{Z}].$$

An object  $M$  of  $\mathcal{O}_{\text{crit}}^{\text{KL}}$  is regarded as a  $R_{\mathcal{Z}}$ -module naturally.

There is a natural map

$$(14) \quad \mathbb{C}[p_0^i; i \in I] \rightarrow \text{Zh}(\mathfrak{Z}(\mathbb{V}_{\mathfrak{g}, \text{crit}})),$$

which is actually an isomorphism. To be precise we have the following assertion.

**Theorem 3.** *The natural map  $\text{Zh}(\mathfrak{Z}(\mathbb{V}_{\mathfrak{g}, \text{crit}})) \rightarrow \text{Zh}(\mathbb{V}_{\mathfrak{g}, \text{crit}}) = U(\bar{\mathfrak{g}})$  is injective and its image coincides with  $\mathcal{Z}(\bar{\mathfrak{g}})$ .*

See [Ara07a] for a proof of Theorem 3. We shall identify  $\text{Zh}(\mathfrak{Z}(\mathbb{V}_{\mathfrak{g}, \text{crit}}))$  with  $\mathcal{Z}(\bar{\mathfrak{g}}) \subset U(\bar{\mathfrak{g}})$  through Theorem 3.

Let  $o(p^i)$  be the image of  $p_0^i$  in  $\text{Zh}(\mathfrak{Z}(\mathbb{V}_{\mathfrak{g}, \text{crit}}))$ . If  $v_\lambda \in M \in \mathcal{O}_{\text{crit}}^{\text{KL}}$  is annihilated by  $\mathfrak{n}_+$ , then one has

$$(15) \quad p_0^i v = o(p^i) v = \bar{\chi}_\lambda(o(p^i)) v,$$

where

$$\bar{\chi}_\lambda : \mathcal{Z}(\bar{\mathfrak{g}}) \rightarrow \mathbb{C}$$

is the evaluation of  $\mathcal{Z}(\bar{\mathfrak{g}})$  at the Verma module of  $\bar{\mathfrak{g}}$  of highest weight  $\bar{\lambda}$ .

Because  $p_0^i$  commutes with the action of  $\mathfrak{g}'$  on  $M \in \mathcal{O}_{\text{crit}}^{\text{KL}}$ , the following assertion follows immediately.

**Proposition 4.** *If  $L(\mu)$  appears in the local composition factor of  $V(\lambda)$  then  $\mu = \lambda - n\delta$  for some  $n \in \mathbb{Z}_{\geq 0}$ .*

**2.6. The conjecture of Frenkel and Gaitsgory for graded opers.** Let  $\bar{\lambda} \in \bar{P}^+$ . The character  $\bar{\chi}_{\bar{\lambda}}$  naturally extends to the graded central character of  $\mathfrak{Z}(\mathbb{V}_{\mathfrak{g}, \text{crit}})$ , that is, to the ring homomorphism

$$(16) \quad \chi_{\bar{\lambda}} : R_{\mathcal{Z}} \rightarrow \mathbb{C}$$

defined by

$$(17) \quad \chi_{\bar{\lambda}}(p_n^i) = \begin{cases} \bar{\chi}_{\bar{\lambda}}(o(p_0^i)) & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

The  $\ker \chi_{\bar{\lambda}} \cdot V(\lambda)$  is a submodule of  $V(\lambda)$ . One has

$$(18) \quad \ker \chi_{\bar{\lambda}} \cdot V(\lambda) = \sum_{n>0, i \in I} U(\mathfrak{n}_-) p_{-n}^i |\lambda\rangle,$$

where  $|\lambda\rangle$  is the highest weight vector of  $V(\lambda)$ . Thus  $\ker \chi_{\bar{\lambda}} \cdot V(\lambda)$  is a proper submodule of  $V(\lambda)$  which lies in  $\mathcal{O}_{\text{crit}}^{\text{KL}}$ . Hence there is a following exact sequence in  $\mathcal{O}_{\text{crit}}^{\text{KL}}$ .

$$(19) \quad V(\lambda) / \ker \chi_{\bar{\lambda}} \cdot V(\lambda) \rightarrow L(\lambda) \rightarrow 0.$$

The following assertion is clear.

**Proposition 5.** *Any vector of  $L(\lambda)$  is annihilated by  $\ker \chi_{\bar{\lambda}}$ .*

Let  $\mathcal{M}_0^{\bar{\lambda}}$  be the full subcategory  $\mathcal{O}_{\text{crit}}^{\text{KL}}$  consisting of objects  $M$  such that  $\ker \chi_{\bar{\lambda}} \cdot M = 0$ . Any simple object of  $\mathcal{M}_0^{\bar{\lambda}}$  is isomorphic to  $L(\mu)$  with  $\mu \in P_{\text{crit}}^+$  such that  $\bar{\mu} = \bar{\lambda}$  (thus all the simple modules are mutually isomorphic as  $\mathfrak{g}'$ -modules).

The following striking assertion was conjectured by Frenkel and Gaitsgory (announced in [Fre06]).

*Conjecture 1* (E. Frenkel and D. Gaitsgory).

- (i) The category  $\mathcal{M}_0^{\bar{\lambda}}$  is semisimple for any  $\bar{\lambda} \in \bar{P}^+$ .
- (ii) For each  $\lambda \in P_{\text{crit}}^+$  there is an isomorphism  $V(\lambda) / \ker \chi_{\bar{\lambda}} \cdot V(\lambda) \cong L(\lambda)$ .

*Remark 6.*

- (i) By the “Langlands duality” [FF92], the  $\chi_{\bar{\lambda}}$  can be considered as an element of the  ${}^L G$ -oper  $\text{Op}_{{}^L G}(D^\times)$  [BD04] on the punctured disk  $D^\times$ , which is “graded”. The original conjecture (Conjecture 5 of [Fre06]) of Frenkel and Gaitsgory is more general and applies to any (not necessarily graded) central character  $\chi$  (i.e. to any element of  $\text{Op}_{{}^L G}^\lambda$ , see Remark 8 and [Fre06]).
- (ii) In the case that  $\bar{\lambda} = 0$ , Conjecture 1 follows from Theorem 6.3 of [FG04] (applied to the graded oper  $\chi_0$ ).

**2.7. Endmorphism rings of Weyl modules.** Let  $\lambda \in P_{\text{crit}}^+$ . Recall that  $|\lambda\rangle$  denotes the highest weight vector of  $V(\lambda)$ . Define

$$(20) \quad R_{\mathcal{Z}}^{\bar{\lambda}} = R_{\mathcal{Z}} / \text{Ann}_{R_{\mathcal{Z}}} |\lambda\rangle.$$

Note that  $R_{\mathcal{Z}}^{\bar{\lambda}}$  is naturally graded by  $D$ :

$$(21) \quad R_{\mathcal{Z}}^{\bar{\lambda}} = \bigoplus_{d \in -\mathbb{Z}_{\geq 0}} (R_{\mathcal{Z}}^{\bar{\lambda}})_d, \quad (R_{\mathcal{Z}}^{\bar{\lambda}})_d = \{a \in R_{\mathcal{Z}}^{\bar{\lambda}}; [D, a] = da\}.$$

There is a natural algebra homomorphism

$$(22) \quad R_{\mathcal{Z}}^{\bar{\lambda}} \rightarrow \text{End}_{U(\mathfrak{g}')} (V(\lambda)).$$

If  $\bar{\lambda} = 0$ , then  $R_{\mathcal{Z}}^0 \cong \mathfrak{Z}(\mathbb{V}_{\mathfrak{g}, \text{crit}})$ . In this case it is known by [FF92, Fre05] that (22) gives an isomorphism

$$(23) \quad \mathfrak{Z}(\mathbb{V}_{\mathfrak{g}, \text{crit}}) \cong \text{End}_{U(\mathfrak{g}')}(\mathbb{V}_{\mathfrak{g}, \text{crit}}).$$

This is true for any  $\lambda \in P_{\text{crit}}^+$ .

**Theorem 7.** *Let  $\lambda \in P_{\text{crit}}^+$ .*

- (i) *The map (22) gives the isomorphism  $R_{\mathcal{Z}}^{\bar{\lambda}} \xrightarrow{\sim} \text{End}_{U(\mathfrak{g}')}(\mathbb{V}(\lambda))$ .*
- (ii) *Set  $\text{ch } R_{\mathcal{Z}}^{\bar{\lambda}} = \bigoplus_{d \in \mathbb{C}} q^d \dim(R_{\mathcal{Z}}^{\bar{\lambda}})_d$ . Then*

$$\text{ch } R_{\mathcal{Z}}^{\bar{\lambda}} = \frac{\prod_{\alpha \in \bar{\Delta}_+} (1 - q^{-\langle \lambda + \rho, \alpha^\vee \rangle})}{\prod_{j \geq 1} (1 - q^{-j})^\ell}.$$

Theorem 7 was obtained earlier in [FG07]. In [Ara07b] we give an independent proof of Theorem 7 by the method of quantum Drinfeld-Sokolov reduction.

*Remark 8.* According to Frenkel and Gaitsgory [FG07], one has

$$\text{Spec } R_{\mathcal{Z}}^{\bar{\lambda}} \cong \text{Op}_{L_G}^{\bar{\lambda}}.$$

where  $\text{Op}_{L_G}^{\bar{\lambda}}$  is a certain sub-pro-variety of  $\text{Op}_{L_G}(D^\times)$  described in [Fre04, FG06] (cf. (7.17) of [Fre06]).

**2.8. An equivalence of categories.** Let  $\mathcal{M}^{\bar{\lambda}}$  be the full subcategory of  $\mathcal{O}_{\text{crit}}^{\text{KL}}$  consisting of objects  $M$  that are annihilated by

$$p_n^i - \chi_{\bar{\lambda}}(p_n^i) \quad \text{for all } i \in I, \ n \geq 0.$$

Then  $V(\lambda), L(\lambda) \in \mathcal{M}^{\bar{\lambda}}$ . Also,  $\mathcal{M}_0^{\bar{\lambda}}$  is a full subcategory of  $\mathcal{M}^{\bar{\lambda}}$ .

Let  $R_{\mathcal{Z}}^{\bar{\lambda}}\text{-grmod}$  be the full subcategory of the category of graded  $R_{\mathcal{Z}}^{\bar{\lambda}}$ -modules consisting of objects  $X = \bigoplus_{d \in \mathbb{C}} X_d$  ( $(R_{\mathcal{Z}}^{\bar{\lambda}})_d \cdot X_{d'} \subset X_{d+d'}$ ) such that (1)  $\dim X_d < \infty$  for all  $d \in \mathbb{C}$ ; (2) there exists a finite subset  $d_1, \dots, d_r \subset \mathbb{C}$  such that  $X_d = 0$  unless  $d \in \bigcup d_i - \mathbb{Z}_{\geq 0}$ . Then any simple object of  $R_{\mathcal{Z}}^{\bar{\lambda}}\text{-grmod}$  is isomorphic to

$$(24) \quad \mathbb{C}_{\chi_{\bar{\lambda}}} := R_{\mathcal{Z}}^{\bar{\lambda}} / \ker \chi_{\bar{\lambda}} \cdot R_{\mathcal{Z}}^{\bar{\lambda}}$$

as  $R_{\mathcal{Z}}^{\bar{\lambda}}$ -modules.

Set

$$(25) \quad F(M) = \text{Hom}_{U(\mathfrak{g}')}(\mathbb{V}(\lambda), M)$$

for  $M \in \mathcal{M}^{\bar{\lambda}}$ . Then  $F(M) \cong M^{n+}$ . The  $F(M)$  is naturally a graded  $R_{\mathcal{Z}}^{\bar{\lambda}}$ -module:

$$(26) \quad F(M) = \bigoplus_{d \in \mathbb{Z}} F(M)_d$$

where  $F(M)_d = M^{n+} \cap M_d$ . Thus  $F$  defines a functor from  $\mathcal{M}^{\bar{\lambda}}$  to  $R_{\mathcal{Z}}^{\bar{\lambda}}\text{-grmod}$ .

Next let

$$(27) \quad G(X) = V(\lambda) \otimes_{R_{\mathcal{Z}}^{\bar{\lambda}}} X$$

for  $X \in R_{\mathcal{Z}}^{\bar{\lambda}}\text{-grmod}$  (See (i) of Theorem 7). Then  $G(X)$  is an object of  $\mathcal{M}^{\bar{\lambda}}$ . Here the action of  $D$  on  $G(X)$  is defined in an obvious way.

**Theorem 9.** *Let  $\lambda \in P_{\text{crit}}^+$ .*

- (i) *The  $V(\lambda)$  is a free  $R_{\mathcal{Z}}^{\bar{\lambda}}$ -module.*

- (ii) (cf. Theorem 6.3 of [FG04]) The functor  $F$  gives an equivalence of categories  $\mathcal{M}^{\bar{\lambda}} \xrightarrow{\sim} R_{\mathbb{Z}}^{\bar{\lambda}}\text{-grmod}$ . The inverse functor is given by  $G$ .

The proof of Theorem 9 is given in [Ara07b].

The following assertion follows immediately from Theorem 9.

**Theorem 10.** *Conjecture 1 holds.*

Because  $L(\lambda) = G(\mathbb{C}_{\chi\bar{\lambda}})$ , by (i) of Theorem 9 one has

$$(28) \quad \text{ch } V(\lambda) = \text{ch } R_{\mathbb{Z}}^{\bar{\lambda}} \cdot \text{ch } L(\lambda).$$

Therefore (6) and (ii) of Theorem 7 give Theorem 1.

*Remark 11.* Using Theorem 9, one can show the irreducibility of the  $\mathfrak{g}'$ -module  $V(\lambda)/\ker \chi \cdot V(\lambda)$  for any  $\chi \in \text{Op}_L^{\bar{\lambda}}G$ . This confirms the original conjecture of Frenkel and Gaitsgory (Conjecture 5 of [Fre06], see Remarks 6 and 8) partially.

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